Unsteady 3D-Navier-Stokes System with Tresca's Friction Law

Mahdi Boukrouche, Imane Boussetouan, Laetitia Paoli

Unsteady 3D-Navier-Stokes System with Tresca's Friction Law

Mahdi Boukrouche*, Imane Boussetouan and Laetitia Paoli †

Abstract

Motivated by extrusion problems, we consider a non-stationary incompressible 3D fluid flow with a non-constant (temperature dependent) viscosity, subjected to mixed boundary conditions with a given time dependent velocity on a part of the boundary and Tresca's friction law on the other part. We construct a sequence of approximate solutions by using a regularization of the free boundary condition due to friction combined with a particular penalty method, reminiscent of the "incompressibility limit" of compressible fluids, allowing to get better insights into the links between the fluid velocity and pressure fields. Then we pass to the limit with compactness arguments to obtain a solution to our original problem.

Keywords: Navier-Stokes system , Tresca's friction law, variational inequality, penalty method.

1 Introduction

Fluid flow problems are involved in several physical phenomena and play an important role in many industrial applications. The fundamental model in fluid mechanics is the well-known Navier-Stokes system for incompressible viscous fluids which has been intensively studied during the last 78 years. Since the pioneering work of J. Leray [14] in 1934, the mathematical analysis of this problem has performed significant progresses: we can mention here only few selected references [16, 11, 22, 6, 9, 10]. Nevertheless it is still a very active research field, from both the theoretical point of view and the numerical point of view (see for instance the very recent research articles [1, 17, 18, 23]).

Motivated by extrusion problems we consider in this paper a non-stationary incompressible 3D fluid flow with a temperature dependent viscosity. As usual for this kind of problems the extrusion device is composed of an upper fixed part and a lower moving part. Several experiments have shown that the classical adhesion condition between the fluid and the lower moving part of the boundary of its domain

*Corresponding author Phone +33 4 77 48 15 35, E-mail: Mahdi.Boukrouche@univ-st-etienne.fr

†Address for all : Lyon University, F-42023 Saint-Etienne, Institut Camille Jordan CNRS UMR 5208, 23 rue Paul Michelon 42023 Saint-Etienne Cedex 2, France. imane.boussetouan@univ-st-etienne.fr, laetitia.paoli@univ-st-etienne.fr

is not satisfied and the real behavior seems to be governed by some friction condition of Tresca's type [12] [24].

More precisely, let ω be a non empty open bounded subset with a Lipschitz continuous boundary, of $\mathbb{R}^{d-1}$ for $d = 2, 3$. We denote by $\Omega \subset \mathbb{R}^d$ the domain of the flow given by

$$\Omega = \{(x', x_d) \in \mathbb{R}^d : \quad x' \in \omega, \quad 0 < x_d < h(x')\},$$

where $x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$, $x = (x', x_d) \in \mathbb{R}^d$. The boundary of Ω is $\partial\Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_1$, where $\Gamma_0 = \{(x', x_d) \in \overline{\Omega} : x_d = 0\}$, $\Gamma_1 = \{(x', x_d) \in \overline{\Omega} : x_d = h(x')\}$ and Γ_L is the lateral boundary. We assume that h is a Lipschitz continuous function and there exist two real numbers h_{min} and h_{max} such that $0 < h_{min} < h(x') < h_{max}$ for all $x' \in \mathbb{R}^{d-1}$.

Let us emphasize that we do not introduce any restrictive assumption on the thickness of the domain. On the contrary to previous papers where only thin films where studied [19, 2, 4], we can consider here general 3D geometries.

The fluid flow is described by the conservation of momentum

$$\frac{\partial v}{\partial t} + (v.\nabla)v = div(\sigma) + f \quad \text{in} \quad \Omega \times (0, \tau),$$

and the incompressibility condition

$$div(v) = 0 \quad \text{in} \quad \Omega \times (0, \tau),$$

where v is the velocity field of the fluid flow, f represents the density of body forces and σ is the stress tensor. We assume that the fluid is Newtonian, so

$$\sigma = -pI + 2\mu(T)D(v),$$

where T depending on $(x, t) \in \Omega \times (0, \tau)$, is the temperature field. Note that T stands for temperature but it will not appear as a variable of the problem, the time interval on which the equations are considered is : $[0, \tau]$. We do this for the main reason that we have generalised our work to a coupled problem (velocity-pressure-temperature) which is in final version, so here we give the regularity of T suitable for our coupled problem [5]. $\mu(T)$ is the temperature dependent viscosity of the fluid, p is the pressure and $D(v)$ is the strain rate tensor given by

$$D(v) = \big(d_{ij}(v)\big)_{1\le i,j\le d}, \quad d_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad 1 \le i, j \le d.$$

Hence v and p satisfy the Navier-Stokes system

$$\frac{\partial v}{\partial t} + (v.\nabla)v - 2div\big(\mu(T)D(v)\big) + \nabla p = f \quad \text{in} \quad \Omega \times (0, \tau), \tag{1}$$

$$div(v) = 0 \quad \text{in} \quad \Omega \times (0, \tau), \tag{2}$$

with the initial condition

$$v(0, \cdot) = v_0 \quad \text{in} \quad \Omega. \tag{3}$$

Let us now describe the boundary conditions. We denote by $s : \Gamma_0 \to \mathbb{R}^{d-1}$ the shear velocity of the lower surface of the extrusion device at $t = 0$ and by $s\zeta(t)$, with $\zeta : [0, \tau] \to \mathbb{R}$ such that $\zeta(0) = 1$, its velocity at any instant $t \in [0, \tau]$. We introduce a function $g : \partial\Omega \to \mathbb{R}^d$ such that

$$\int_{\Gamma_L} g.n \, d\sigma = 0, \quad g = 0 \text{ on } \Gamma_1, \quad g_n = g \cdot n = 0 \text{ and } g_\tau = g - g_n n = s \text{ on } \Gamma_0,$$

where $n = (n_1, \ldots, n_d)$ is the unit outward normal vector to $\partial\Omega$. We denote here by $u \cdot w$ the Euclidean inner product of two vectors u and w and by $|.|$ the Euclidian norm. We define respectively the normal and the tangential velocities on Γ_0 by

$$v_n = v \cdot n = v_i n_i, \quad v_{\mathcal{T}} = \big(v_{\mathcal{T}i}\big)_{1 \le i \le d} \text{ with } v_{\mathcal{T}i} = v_i - v_n n_i \quad 1 \le i \le d$$

and the normal and the tangential components of the stress tensor on Γ_0 by

$$\sigma_n = (\sigma \cdot n) \cdot n = \sigma_{ij} n_i n_j, \quad \sigma_{\mathcal{T}} = \big(\sigma_{\mathcal{T}i}\big)_{1 \le i \le d} \text{ with } \sigma_{\mathcal{T}i} = \sigma_{ij} n_j - \sigma_n n_i \quad 1 \le i \le d.$$

Note that we will use the Einstein's summation convention throughout this paper. We assume that the upper surface of the extrusion device is fixed i.e.

$$v = 0 \quad \text{on} \quad \Gamma_1 \times (0, \tau), \tag{4}$$

the given velocity on the lateral boundary is the product $g(x)\zeta(t)$ i.e.

$$v = g\zeta \quad \text{on} \quad \Gamma_L \times (0, \tau), \tag{5}$$

and the normal component of the velocity on the lower part of boundary is given by

$$v_n = v \cdot n = 0 \quad \text{on} \quad \Gamma_0 \times (0, \tau). \tag{6}$$

The tangential velocity on $\Gamma_0 \times (0, \tau)$ is unknown and satisfies Tresca's friction law [8]

$$\begin{array}{l} |\sigma_{\mathcal{T}}| < \ell \Rightarrow v_{\mathcal{T}} = (s\zeta, 0) \\ |\sigma_{\mathcal{T}}| = \ell \Rightarrow \exists \lambda \ge 0 \quad \text{such that} \quad v_{\mathcal{T}} = (s\zeta, 0) - \lambda \sigma_{\mathcal{T}} \end{array} \tag{7}$$

where $\ell : [0, \tau] \times \Gamma_0 \to \mathbb{R}$ is the upper limit for the shear stress (i.e. ℓ is the Tresca's friction threshold).

The paper is organized as follows. In Section 2 we introduce the functional framework and the formulation of the problem as a variational inequality for the fluid velocity and pressure fields. In Section 3 we use a regularization of Tresca's functional to obtain a sequence of approximate problems (P_ε) of Navier-Stokes type.

A classical technique to study such problems is to choose divergence free test-functions in order to "kill" the pressure terms then to solve the derived variational problem for the fluid velocity and to get finally the pressure by applying abstract results of functional analysis (see [22, 21, 10] for instance). The major drawback of this technique is that the pressure appears as a by product. In order to get better insights into the links between the velocity and pressure fields, we adopt in this paper another approach, reminiscent of the "incompressiblity limit" of compressible fluids.

More precisely, following an idea of J.L. Lions [15], we relax the divergence free condition and we propose a sequence of penalized problems (P_ε^δ). In Section 4 we establish the existence of solutions to this family of problems $(P_\varepsilon^\delta)_{\varepsilon>0,\delta>0}$ and we obtain some a priori estimates. Next in Section 5 we define a sequence of approximate pressures $(p_\varepsilon^\delta)_{\varepsilon>0,\delta>0}$ and we study its properties. By using functional spaces that are weaker in time than in space, we succeed in obtaining good enough uniform estimates with respect to the the parameters δ and ε. Then in Section 6 we use compactness arguments to pass to the incompressible limit as δ tends to zero and we show that the limit velocity and pressure fields are solutions to the problems (P_ε). Finally we pass to the limit as ε tends to zero and we get a solution to our original variational inequality.

2 Variational formulation of the problem

We denote by

$$\mathbf{H}^1(\Omega) = \big(H^1(\Omega)\big)^d,\ \mathbf{L}^2(\Omega) = (L^2(\Omega))^d,\ \mathbf{H}^{\frac{1}{2}}(\partial\Omega) = (H^{\frac{1}{2}}(\partial\Omega))^d,\ \mathbf{H}^2(\Omega) = \big(H^2(\Omega)\big)^d.$$

We assume that

$$\begin{array}{l} f \in L^2\big(0,\tau;\mathbf{L}^2(\Omega)\big),\ \ell \in L^2\big(0,\tau;\mathbf{L}^2_+(\Gamma_0)\big) \cap L^\infty\big(0,\tau;\mathbf{L}^\infty_+(\Gamma_0)\big), \\ \zeta \in \mathcal{C}^\infty\big([0,\tau]\big) \text{ with } \zeta(0) = 1, \end{array} \tag{8}$$

with $\mathbf{L}^2_+(\Gamma_0) = \mathbf{L}^2(\Gamma_0;\mathbb{R}^+)$ (respectively $\mathbf{L}^\infty_+(\Gamma_0) = \mathbf{L}^\infty(\Gamma_0;\mathbb{R}^+)$ and $\tau > 0$. The viscosity $\mu(T)$ is a function of $L^\infty\big(0,\tau;L^\infty(\Omega)\big)$ depending on the temperature T, and there exist two real numbers μ^*, μ_* such that

$$0 < \mu^* \leq 2\mu(X) \leq \mu_* \quad \forall X \in \mathbb{R}, \tag{9}$$

and also there exists an extension of g to Ω, denoted by G_0, such that

$$\begin{array}{l} G_0 \in \mathbf{H}^2(\Omega),\ div(G_0) = 0 \text{ in } \Omega,\ G_0 = g \text{ on } \Gamma_L,\ G_0 = 0 \text{ on } \Gamma_1, \\ G_{0n} = 0 \text{ and } G_{0\tau} = s \text{ on } \Gamma_0. \end{array} \tag{10}$$

We introduce now the following functional framework

$$\mathcal{V}_0 = \big\{\varphi \in \mathbf{H}^1(\Omega) :\ \varphi = 0 \text{ on } \Gamma_L \cup \Gamma_1,\ \varphi_n = 0 \text{ on } \Gamma_0\big\},$$

endowed with the norm of $\mathbf{H}^1(\Omega)$ and

$$\mathcal{V}_{0div} = \big\{\varphi \in \mathcal{V}_0 :\ div(\varphi) = 0 \text{ in } \Omega\big\}.$$

Moreover let

$$L^2_0(\Omega) = \left\{q \in L^2(\Omega) :\ \int_\Omega q\, dx = 0\right\}.$$

We define the following applications

$$\begin{array}{rcl} a(T;\cdot,\cdot) : L^2(0,\tau;\mathcal{V}_0) \times L^2(0,\tau;\mathcal{V}_0) & \to & \mathbb{R} \\ (u,v) & \mapsto & a(T;u,v) = \displaystyle\int_0^\tau \int_\Omega 2\mu(T) D(u) : D(v)\, dxdt \\ & & \qquad = \displaystyle\int_0^\tau \int_\Omega 2\mu(T) d_{ij}(u) d_{ij}(v)\, dxdt, \end{array}$$

and

$$\begin{array}{rrcl}\Psi: & L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big) & \to & \mathbb{R}\\ & u & \mapsto & \Psi(u)=\displaystyle\int_0^\tau\int_{\Gamma_0}\ell|u|\,dx'dt.\end{array}$$

We may observe that Ψ is convex continuous but not differentiable.

Let b be the usual trilinear form given by

$$\begin{array}{rrcl}b: & \mathcal{V}_0\times\mathcal{V}_0\times\mathcal{V}_0 & \to & \mathbb{R}\\ & (u,v,w) & \mapsto & b(u,v,w)=\displaystyle\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j\,dx.\end{array}$$

By definition of $\mathcal{V}_0$ we have the identity

$$b(u,v,w)=-b(u,w,v)-\int_\Omega div(u)v\cdot w\,dx\quad\forall(u,v,w)\in\mathcal{V}_0\times\mathcal{V}_0\times\mathcal{V}_0. \tag{11}$$

Moreover, using Korn's inequality [13] and assumption (9), there exists $\alpha>0$ such that, for almost every $t\in(0,\tau)$, we have

$$\alpha\|u\|^2_{\mathbf{H}^1(\Omega)}\le\int_\Omega 2\mu(T)D(u):D(u)\,dx\le\mu_*\|u\|^2_{\mathbf{H}^1(\Omega)}\quad\forall u\in\mathcal{V}_0. \tag{12}$$

In order to deal with homogeneous boundary conditions on $\Gamma_L\cup\Gamma_1$, we set $\widetilde{v}=v-G_0\zeta$. The variational formulation of the problem (1)-(7) is given by (see for example [8] and [2, 4])

Problem (P) Find

$$\widetilde{v}\in L^2\big(0,\tau;\mathcal{V}_{0div}\big)\cap L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big),\ \frac{\partial\widetilde{v}}{\partial t}\in L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big),\ p\in H^{-1}\big(0,\tau;L^2_0(\Omega)\big)$$

such that, for all $\varphi\in\mathcal{V}_0$ and for all $\chi\in\mathcal{D}(0,\tau)$, we have

$$\begin{aligned}&\left\langle\frac{d}{dt}(\widetilde{v},\varphi),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}+\big\langle b(\widetilde{v},\widetilde{v},\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}-\big\langle\big(p,div(\varphi)\big),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}\\&+a(T;\widetilde{v},\varphi\chi)+\Psi(\widetilde{v}+\varphi\chi)-\Psi(\widetilde{v})\ge\big\langle(f,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}-a(T;G_0\zeta,\varphi\chi)\\&-\left\langle\left(G_0\frac{\partial\zeta}{\partial t},\varphi\right),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}-\big\langle b(G_0\zeta,\widetilde{v}+G_0\zeta,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}\\&-\big\langle b(\widetilde{v},G_0\zeta,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}\end{aligned} \tag{13}$$

with the initial condition

$$\widetilde{v}(0,\cdot)=\widetilde{v}_0\in\mathbf{H}, \tag{14}$$

where $\mathbf{H}$ is the well known closure in $\mathbf{L}^2(\Omega)$ of the space

$$\{\varphi\in\mathcal{C}^\infty(\overline{\Omega}):\ div\,\varphi=0\text{ in }\Omega\},$$

$(\cdot,\cdot)$ denotes the inner product in $\mathbf{L}^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}$ the duality product between $\mathcal{D}'(0,\tau)$ and $\mathcal{D}(0,\tau)$. Let us emphasize that we identify $\widetilde{v}+\varphi\chi$ and $\widetilde{v}$ with their trace on Γ_0 in the definition of $\Psi(\widetilde{v}+\varphi\chi)$ and $\Psi(\widetilde{v})$.

3 Approximate problems

The variational formulation of the problem (1)-(7) leads to an inequality involving Tresca's functional Ψ, which is convex continuous but not differentiable. To overcome this difficulty we use a regularization of Ψ. More precisley, for any $\varepsilon > 0$, we introduce Ψ_ε defined by

$$\Psi_\varepsilon(u) = \int_0^\tau \int_{\Gamma_0} \ell\sqrt{\varepsilon^2 + |u|^2}\, dx'dt \quad \forall u \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$$

which is Gâteaux-differentiable in $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$, with $\Psi'_\varepsilon(u) \in \big(L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)\big)' = L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$ for all $u \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$ given by

$$\langle \Psi'_\varepsilon(u), w\rangle = \int_0^\tau \int_{\Gamma_0} \ell\frac{u\cdot w}{\sqrt{\varepsilon^2+|u|^2}}\, dx'dt \quad \forall w \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$. Then we consider a sequence of initial data $(\widetilde{v}_{\varepsilon 0})_{\varepsilon>0}$ such that

$$\widetilde{v}_{\varepsilon 0} \longrightarrow_{\varepsilon\to 0} \widetilde{v}_0 \quad \text{strongly in } \mathbf{H} \tag{15}$$

and we approximate problem (P) by the following problems (P_ε), $\varepsilon > 0$:
Problem (P_ε) Find

$$\widetilde{v}_\varepsilon \in L^2\big(0,\tau;\mathcal{V}_{0div}\big)\cap L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big),\ \frac{\partial \widetilde{v}_\varepsilon}{\partial t} \in L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big),\ p_\varepsilon \in H^{-1}\big(0,\tau;L^2_0(\Omega)\big)$$

such that, for all $\varphi \in \mathcal{V}_0$ and for all $\chi \in \mathcal{D}(0,\tau)$, we have

$$\begin{aligned}
&\Big\langle \frac{d}{dt}(\widetilde{v}_\varepsilon,\varphi),\chi\Big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} + \big\langle b(\widetilde{v}_\varepsilon,\widetilde{v}_\varepsilon,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - \big\langle\big(p_\varepsilon, div(\varphi)\big),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}\\
&+a(T;\widetilde{v}_\varepsilon,\varphi\chi) + \big\langle\Psi'_\varepsilon(\widetilde{v}_\varepsilon),\varphi\chi\big\rangle = \big\langle(f,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - a(T;G_0\zeta,\varphi\chi)\\
&-\Big\langle\Big(G_0\frac{\partial\zeta}{\partial t},\varphi\Big),\chi\Big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - \big\langle b(G_0\zeta,\widetilde{v}_\varepsilon + G_0\zeta,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}\\
&-\big\langle b(\widetilde{v}_\varepsilon,G_0\zeta,\varphi),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}
\end{aligned} \tag{16}$$

with the initial condition

$$\widetilde{v}_\varepsilon(0,\cdot) = \widetilde{v}_{\varepsilon 0} \in \mathbf{H}. \tag{17}$$

As it has been explained in Section 1, a classical technique to solve such problems consists in choosing divergence free test-functions. Indeed if $\varphi \in \mathcal{V}_{0div}$, the term $\big\langle\big(p_\varepsilon, div(\varphi)\big),\chi\big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}$ vanishes and we simply get a variational problem for the fluid velocity $\widetilde{v}_\varepsilon$. Then the existence of $p_\varepsilon \in H^{-1}\big(0,\tau;L^2_0(\Omega)\big)$ is derived via abstract results of functional analysis (see [22, 21, 10] for instance).

With this technique the pressure appears as a by product of the study. In order to get better insights into the links between the velocity and pressure fields, we will follow an idea proposed by J.L. Lions in [15] and recently used in [3], which consists in relaxing the divergence free condition. More precisely, we consider the following penalized problems (P_ε^δ), $\delta > 0$, $\varepsilon > 0$:

Problem (P_ε^δ) Find

$$\widetilde{v}_\varepsilon^\delta \in L^2\big(0,\tau;\mathcal{V}_0\big) \cap L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big), \quad \frac{\partial \widetilde{v}_\varepsilon^\delta}{\partial t} \in L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_0)'\big)$$

such that, for all $\varphi \in \mathcal{V}_0$ and for all $\chi \in \mathcal{D}(0,\tau)$, we have

$$\begin{aligned}
&\left\langle \frac{d}{dt}\left(\widetilde{v}_\varepsilon^\delta, \varphi\right), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} + \left\langle b(\widetilde{v}_\varepsilon^\delta, \widetilde{v}_\varepsilon^\delta, \varphi), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} \\
&+\frac{1}{2}\left\langle \int_\Omega \widetilde{v}_\varepsilon^\delta div(\widetilde{v}_\varepsilon^\delta)\varphi\,dx, \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} + \frac{1}{\delta}\left\langle \big(div(\widetilde{v}_\varepsilon^\delta), div(\varphi)\big), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} + a(T;\widetilde{v}_\varepsilon^\delta, \varphi\chi) \\
&+\left\langle \Psi_\varepsilon'(\widetilde{v}_\varepsilon^\delta), \varphi\chi\right\rangle = \left\langle (f,\varphi), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - a(T;G_0\zeta, \varphi\chi) - \left\langle \left(G_0\frac{\partial\zeta}{\partial t}, \varphi\right), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} \\
&-\left\langle b(G_0\zeta, \widetilde{v}_\varepsilon^\delta + G_0\zeta, \varphi), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - \left\langle b(\widetilde{v}_\varepsilon^\delta, G_0\zeta, \varphi), \chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}
\end{aligned} \tag{18}$$

with the initial condition

$$\widetilde{v}_\varepsilon^\delta(0,\cdot) = \widetilde{v}_{\varepsilon 0}^\delta \in \mathbf{L}^2(\Omega) \tag{19}$$

and we assume that the sequence of initial data $(\widetilde{v}_{\varepsilon 0}^\delta)_{\delta>0}$ satisfies

$$\widetilde{v}_{\varepsilon 0}^\delta \longrightarrow_{\delta\to 0} \widetilde{v}_{\varepsilon 0} \quad \text{strongly in } \mathbf{L}^2(\Omega). \tag{20}$$

Let us emphasize that the last term of the first line of (18) is added for technical reasons (see (28)) while the first term of the second line is the penalty term: $-\frac{1}{\delta}div(\widetilde{v}_\varepsilon^\delta)$ will play the role of an approximate pressure (see Section 5). Furthermore, the approximate initial velocities $(\widetilde{v}_{\varepsilon 0}^\delta)_{\varepsilon>0,\delta>0}$ and $(\widetilde{v}_{\varepsilon 0})_{\varepsilon>0}$ are not assumed to be more regular that $\widetilde{v}_0$.

4 Existence result for the penalized problems (P_ε^δ)

We prove the existence of solutions for the system (18)-(19), for any $\varepsilon > 0$ and $\delta > 0$, by using the Galerkin method. Since $\mathcal{V}_0$ is a closed subspace of $\mathbf{H}^1(\Omega)$, it admits an Hilbertian basis $(w_i)_{i\geq 1}$, which is orthogonal for the inner product of $\mathbf{H}^1(\Omega)$ and orthonormal for the inner product of $\mathbf{L}^2(\Omega)$. Then, for all $m \geq 1$, we look for a function $\widetilde{v}_{\varepsilon m}^\delta$ given by

$$\widetilde{v}_{\varepsilon m}^\delta(t,x) = \sum_{j=1}^m g_{\varepsilon j}^\delta(t) w_j(x), \quad \forall t \in (0,\tau),\ \forall x \in \Omega, \tag{21}$$

such that, for all $k \in \{1, \dots, m\}$, we have

$$\begin{aligned}
&\left(\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t}, w_k\right) + b(\widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}, w_k) + \frac{1}{2}\int_{\Omega} \widetilde{v}^{\delta}_{\varepsilon m} div(\widetilde{v}^{\delta}_{\varepsilon m}) w_k \, dx + \frac{1}{\delta}\Big(div(\widetilde{v}^{\delta}_{\varepsilon m}), div(w_k)\Big) \\
&+ \int_{\Omega} 2\mu(T) D(\widetilde{v}^{\delta}_{\varepsilon m}) : D(w_k)\, dx + \int_{\Gamma_0} \ell \frac{\widetilde{v}^{\delta}_{\varepsilon m} \cdot w_k}{\sqrt{\varepsilon^2 + |\widetilde{v}^{\delta}_{\varepsilon m}|^2}}\, dx' = (f, w_k) \\
&- \int_{\Omega} 2\mu(T) D(G_0\zeta) : D(w_k)\, dx - \left(G_0 \frac{\partial \zeta}{\partial t}, w_k\right) - b(G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m} + G_0\zeta, w_k) \\
&- b(\widetilde{v}^{\delta}_{\varepsilon m}, G_0\zeta, w_k) \quad \text{a.e. in } (0,\tau)
\end{aligned} \tag{22}$$

with the initial condition

$$\widetilde{v}^{\delta}_{\varepsilon m}(0, \cdot) = \widetilde{v}^{\delta}_{\varepsilon m0} \tag{23}$$

where $\widetilde{v}^{\delta}_{\varepsilon m0}$ is defined as the orthogonal projection of $\widetilde{v}^{\delta}_{\varepsilon 0}$ in $\mathbf{L}^2(\Omega)$ on $\mathrm{Span}\big\{w_1 \dots w_m\big\}$. For all $i, j, k \in \{1, \dots, m\}$ we denote

$$F_k = (f, w_k) - \int_{\Omega} 2\mu(T) D(G_0\zeta) : D(w_k)\, dx - \left(G_0 \frac{\partial \zeta}{\partial t}, w_k\right) - b(G_0\zeta, G_0\zeta, w_k) \in L^2(0,\tau)$$

and

$$A_{j,k}(T) = \int_{\Omega} 2\mu(T) D(w_j) : D(w_k)\, dx \in L^{\infty}(0,\tau), \quad B_{i,j,k} = b(w_i, w_j, w_k) \in \mathbb{R}.$$

By replacing $\widetilde{v}^{\delta}_{\varepsilon m}$ by its expression (21) in equation (22) and using the orthonormality of $(w_i)_{i\geq 1}$ in $\mathbf{L}^2(\Omega)$, we obtain

$$\begin{aligned}
&(g^{\delta}_{\varepsilon k})' + \sum_{i,j=1}^{m} g^{\delta}_{\varepsilon j} g^{\delta}_{\varepsilon i} B_{i,j,k} + \frac{1}{2}\sum_{i,j=1}^{m} g^{\delta}_{\varepsilon i} g^{\delta}_{\varepsilon j} \int_{\Omega} w_i div(w_j) w_k\, dx + \frac{1}{\delta}\sum_{j=1}^{m} g^{\delta}_{\varepsilon j}\big(div(w_j), div(w_k)\big) \\
&+ \sum_{j=1}^{m} g^{\delta}_{\varepsilon j} A_{j,k}(T) + \int_{\Gamma_0} \ell \frac{\big(\sum_{j=1}^{m} g^{\delta}_{\varepsilon j} w_j\big)\cdot w_k}{\sqrt{\varepsilon^2 + \big|\sum_{j=1}^{m} g^{\delta}_{\varepsilon j} w_j\big|^2}}\, dx' = F_k \\
&- \sum_{j=1}^{m} g^{\delta}_{\varepsilon j} b(G_0\zeta, w_j, w_k) - \sum_{j=1}^{m} g^{\delta}_{\varepsilon j} b(w_j, G_0\zeta, w_k) \quad \forall k \in \{1, \dots, m\}.
\end{aligned} \tag{24}$$

We can rewrite this differential system as

$$(g^{\delta}_{\varepsilon})' = \mathcal{G}(t, g^{\delta}_{\varepsilon}), \quad g^{\delta}_{\varepsilon} = \big(g^{\delta}_{\varepsilon j}\big)_{1\leq j\leq m}$$

where $\mathcal{G}$ satisfies the assumptions of the Caratheodory theorem (see [7]). Moreover, the function $\mathcal{G}$ is locally Lipschitz continuous with respect its the second argument. It follows that, for any given initial data, the differential system (24) admits an unique maximal solution $g^{\delta}_{\varepsilon j}$ in $H^1(0,\tau_m)$, $1 \leq j \leq m$, with $0 < \tau_m \leq \tau$, which implies the existence of a maximal solution $\widetilde{v}^{\delta}_{\varepsilon m} \in H^1(0,\tau_m;\mathcal{V}_0)$ to (22)-(23). In the following lemma, some a priori estimates independent of m, δ and ε will be established, which allow us to extend this solution to the whole interval $[0,\tau]$.

Lemma 4.1. *Assume that (8), (9) and (10) hold and that* $(\widetilde{v}^{\delta}_{\varepsilon 0})_{\varepsilon>0,\delta>0}$ *is a bounded sequence of* $\mathbf{L}^2(\Omega)$*. The problem (22)-(23) admits a unique solution* $\widetilde{v}^{\delta}_{\varepsilon m} \in H^1\big(0,\tau;\mathcal{V}_0\big)$ *which satisfies the following estimates*

$$\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))} \leq C \tag{25}$$

$$\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{L^2(0,\tau;\mathbf{H}^1(\Omega))} \leq C \tag{26}$$

$$\|div(\widetilde{v}^{\delta}_{\varepsilon m})\|_{L^2(0,\tau;L^2(\Omega))} \leq C\sqrt{\delta} \tag{27}$$

where C *is a constant independent of* m*,* δ *and* ε*.*

Proof. By multiplying equation (22) by $g^{\delta}_{\varepsilon k}(t)$ and adding from $k = 1$ to m, we obtain

$$\begin{aligned}
&\left(\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t}, \widetilde{v}^{\delta}_{\varepsilon m}\right) + b(\widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}) + \frac{1}{2}\int_\Omega \widetilde{v}^{\delta}_{\varepsilon m} div(\widetilde{v}^{\delta}_{\varepsilon m})\widetilde{v}^{\delta}_{\varepsilon m}\, dx + \frac{1}{\delta}\Big(div(\widetilde{v}^{\delta}_{\varepsilon m}), div(\widetilde{v}^{\delta}_{\varepsilon m})\Big) \\
&+ \int_\Omega 2\mu(T)D(\widetilde{v}^{\delta}_{\varepsilon m}) : D(\widetilde{v}^{\delta}_{\varepsilon m})\, dx + \int_{\Gamma_0} \ell \frac{|\widetilde{v}^{\delta}_{\varepsilon m}|^2}{\sqrt{\varepsilon^2 + |\widetilde{v}^{\delta}_{\varepsilon m}|^2}}\, dx' = (f, \widetilde{v}^{\delta}_{\varepsilon m}) \\
&- \int_\Omega 2\mu(T)D(G_0\zeta) : D(\widetilde{v}^{\delta}_{\varepsilon m})\, dx - \left(G_0\frac{\partial \zeta}{\partial t}, \widetilde{v}^{\delta}_{\varepsilon m}\right) - b(G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m} + G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m}) \\
&- b(\widetilde{v}^{\delta}_{\varepsilon m}, G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m}) \quad \text{a.e. in } (0, \tau_m).
\end{aligned}$$

With (11) we get

$$b(\widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}) + \frac{1}{2}\int_\Omega \widetilde{v}^{\delta}_{\varepsilon m} div(\widetilde{v}^{\delta}_{\varepsilon m})\widetilde{v}^{\delta}_{\varepsilon m}\, dx = 0 \tag{28}$$

and since $div(G_0) = 0$ in Ω, we have also $b(G_0, \widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}) = 0$. Furthermore since $\ell \in L^2(0,\tau;\mathbf{L}^2_+(\Gamma_0))$, we obtain

$$\begin{aligned}
&\left(\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t}, \widetilde{v}^{\delta}_{\varepsilon m}\right) + \frac{1}{\delta}\Big(div(\widetilde{v}^{\delta}_{\varepsilon m}), div(\widetilde{v}^{\delta}_{\varepsilon m})\Big) + \int_\Omega 2\mu(T)D(\widetilde{v}^{\delta}_{\varepsilon m}) : D(\widetilde{v}^{\delta}_{\varepsilon m})\, dx \\
&\leq (f, \widetilde{v}^{\delta}_{\varepsilon m}) - \int_\Omega 2\mu(T)D(G_0\zeta) : D(\widetilde{v}^{\delta}_{\varepsilon m})\, dx - \left(G_0\frac{\partial \zeta}{\partial t}, \widetilde{v}^{\delta}_{\varepsilon m}\right) \\
&- b(G_0\zeta, G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m}) - b(\widetilde{v}^{\delta}_{\varepsilon m}, G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m}) \quad \text{a.e. in } (0, \tau_m).
\end{aligned}$$

Let us estimate now the terms in the right-hand side of the previous inequality. We denote hereinafter by K the constant of the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$. By using Cauchy-Schwarz's and Young's inequalities, we obtain

$$\begin{aligned}
\Big|(f, \widetilde{v}^{\delta}_{\varepsilon m})\Big| &\leq \|f\|_{\mathbf{L}^2(\Omega)}\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^2(\Omega)} \\
&\leq \frac{1}{2}\|f\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}$$

$$\begin{aligned}
\left|\int_\Omega 2\mu(T)D(G_0\zeta) : D(\widetilde{v}^{\delta}_{\varepsilon m})\, dx\right| &\leq \mu_*|\zeta|\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{H}^1(\Omega)}\|G_0\|_{\mathbf{H}^1(\Omega)} \\
&\leq \frac{\alpha}{4}\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{H}^1(\Omega)} + \frac{\mu_*^2}{\alpha}|\zeta|^2\|G_0\|^2_{\mathbf{H}^1(\Omega)},
\end{aligned}$$

$$
\begin{aligned}
\left|\left(G_0\frac{\partial\zeta}{\partial t},\widetilde{v}^{\delta}_{\varepsilon m}\right)\right| &\leq \left|\frac{\partial\zeta}{\partial t}\right|\|G_0\|_{\mathbf{L}^2(\Omega)}\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^2(\Omega)}\\
&\leq \frac{1}{2}\left|\frac{\partial\zeta}{\partial t}\right|^2\|G_0\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{2}\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)},
\end{aligned}
$$

$$
\begin{aligned}
\left|b(G_0\zeta,G_0\zeta,\widetilde{v}^{\delta}_{\varepsilon m})\right| &\leq |\zeta|^2\|G_0\|_{\mathbf{L}^4(\Omega)}\|\nabla G_0\|_{\mathbf{L}^4(\Omega)}\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^2(\Omega)}\\
&\leq \frac{1}{2}\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)}+\frac{K^4}{2}|\zeta|^4\|G_0\|^2_{\mathbf{H}^1(\Omega)}\|\nabla G_0\|^2_{\mathbf{H}^1(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|b(\widetilde{v}^{\delta}_{\varepsilon m},G_0\zeta,\widetilde{v}^{\delta}_{\varepsilon m})\right| &\leq |\zeta|\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^4(\Omega)}\|\nabla G_0\|_{\mathbf{L}^4(\Omega)}\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^2(\Omega)}\\
&\leq \frac{\alpha}{4}\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{H}^1(\Omega)}+\frac{K^4}{\alpha}|\zeta|^2\|\nabla G_0\|^2_{\mathbf{H}^1(\Omega)}\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)}.
\end{aligned}
$$

With (12) and an integration from 0 to s, with $0<s<\tau_m$, we get

$$
\begin{aligned}
&\frac{1}{2}\|\widetilde{v}^{\delta}_{\varepsilon m}(s)\|^2_{\mathbf{L^2(\Omega)}}+\frac{1}{\delta}\int_0^s\|div(\widetilde{v}^{\delta}_{\varepsilon m})\|^2_{L^2(\Omega)}\,dt+\frac{\alpha}{2}\int_0^s\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{H}^1(\Omega)}\,dt\leq\frac{1}{2}\|\widetilde{v}^{\delta}_{\varepsilon m}(0)\|^2_{\mathbf{L}^2(\Omega)}\\
&+\frac{1}{2}\int_0^s\|f\|^2_{\mathbf{L}^2(\Omega)}\,dt+\frac{\mu_*^2}{\alpha}\|G_0\|^2_{\mathbf{H}^1(\Omega)}\int_0^s|\zeta|^2\,dt+\frac{1}{2}\|G_0\|^2_{\mathbf{L}^2(\Omega)}\int_0^s\left|\frac{\partial\zeta}{\partial t}\right|^2\,dt\\
&+\frac{3}{2}\int_0^s\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)}\,dt+\frac{K^4}{\alpha}\|\nabla G_0\|^2_{\mathbf{H}^1(\Omega)}\int_0^s|\zeta|^2\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)}\,dt\\
&+\frac{K^4}{2}\|G_0\|^2_{\mathbf{H}^1(\Omega)}\|\nabla G_0\|^2_{\mathbf{H}^1(\Omega)}\int_0^s|\zeta|^4\,dt.
\end{aligned}
$$

Reminding that $\widetilde{v}^{\delta}_{\varepsilon m0}$ is defined as the orthogonal projection of $\widetilde{v}^{\delta}_{\varepsilon 0}$ in $\mathbf{L}^2(\Omega)$ on $\mathrm{Span}\{w_1\dots w_m\}$ and that the sequence $(\widetilde{v}^{\delta}_{\varepsilon 0})_{\varepsilon>0,\delta>0}$ is bounded in $\mathbf{L}^2(\Omega)$, we infer that there exists a constant C_0, independent of δ and ε such that

$$
\|\widetilde{v}^{\delta}_{\varepsilon m}(0)\|_{L^2(\Omega)}=\|\widetilde{v}^{\delta}_{\varepsilon m0}\|_{L^2(\Omega)}\leq\|\widetilde{v}^{\delta}_{\varepsilon 0}\|_{L^2(\Omega)}\leq C_0\quad\forall m\geq 1,\ \forall\delta>0,\ \forall\varepsilon>0.
$$

It follows that

$$
\begin{aligned}
&\frac{1}{2}\|\widetilde{v}^{\delta}_{\varepsilon m}(s)\|^2_{\mathbf{L}^2(\Omega)}+\frac{1}{\delta}\int_0^s\|div(\widetilde{v}^{\delta}_{\varepsilon m})\|^2_{L^2(\Omega)}\,dt+\frac{\alpha}{2}\int_0^s\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{H}^1(\Omega)}\,dt\leq C_1\\
&+C_2\int_0^s\|\widetilde{v}^{\delta}_{\varepsilon m}\|^2_{\mathbf{L}^2(\Omega)}\,dt,
\end{aligned}
\tag{29}
$$

where C_1 and C_2 are two constants independent of m, δ and ε, namely

$$
\begin{aligned}
C_1=&\frac{1}{2}C_0^2+\frac{1}{2}\int_0^\tau\|f\|^2_{\mathbf{L}^2(\Omega)}\,dt+\frac{\mu_*^2}{\alpha}\|G_0\|^2_{\mathbf{H}^1(\Omega)}\int_0^\tau|\zeta|^2\,dt+\frac{1}{2}\|G_0\|^2_{\mathbf{L}^2(\Omega)}\int_0^\tau\left|\frac{\partial\zeta}{\partial t}\right|^2\,dt\\
&+\frac{K^4}{2}\|G_0\|^2_{\mathbf{H}^1(\Omega)}\|\nabla G_0\|^2_{\mathbf{H}^1(\Omega)}\int_0^\tau|\zeta|^4\,dt
\end{aligned}
$$

and

$$
C_2=\frac{3}{2}+\frac{K^4}{\alpha}\|\nabla G_0\|^2_{\mathbf{H}^1(\Omega)}\|\zeta\|^2_{L^\infty(0,\tau)}.
$$

With Grönwall's lemma, we obtain

$$\|\widetilde{v}^{\delta}_{\varepsilon m}(s)\|^2_{\mathbf{L}^2(\Omega)} \leq 2C_1 \exp(2sC_2) \leq 2C_1 \exp(2\tau C_2) \quad \forall s \in [0, \tau_m). \tag{30}$$

With (21) and (24) we infer that the functions $g^{\delta}_{\varepsilon j}$, $1 \leq j \leq m$, admit a limit at τ_m and, by definition of the maximal solution, we may conclude that $\tau_m = \tau$. Now, (25) follows from (30). By inserting (30) in (29) with $s = \tau$, we obtain (26) and (27). $\square$ $\square$

In the following lemma, we establish an estimate of the time derivative for the approximate velocity.

Lemma 4.2. *Under the same assumptions as in Lemma 4.1, we have*

$$\left\| \frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t} \right\|_{L^{\frac{4}{3}}(0,\tau;\mathcal{V}'_0)} \leq C_\delta \tag{31}$$

where C_δ is a constant independent of m and ε.

Proof. Let $\varphi \in \mathcal{V}_0$. For all $m \geq 1$, we define φ_m as the othogonal projection with respect to the inner product of $\mathbf{H}^1(\Omega)$ of φ on $\mathrm{Span}\{w_1, \ldots, w_m\}$. With (22) we get

$$\begin{aligned}
&\left(\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t}, \varphi_m \right) = -b(\widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}, \varphi_m) - \frac{1}{2} \int_\Omega \widetilde{v}^{\delta}_{\varepsilon m} div(\widetilde{v}^{\delta}_{\varepsilon m}) \varphi_m \, dx - \frac{1}{\delta} \big(div(\widetilde{v}^{\delta}_{\varepsilon m}), div(\varphi_m) \big) \\
&- \int_\Omega 2\mu(T) D(\widetilde{v}^{\delta}_{\varepsilon m}) : D(\varphi_m) \, dx - \int_{\Gamma_0} \ell \frac{\widetilde{v}^{\delta}_{\varepsilon m} \cdot \varphi_m}{\sqrt{\varepsilon^2 + |\widetilde{v}^{\delta}_{\varepsilon m}|^2}} \, dx' + (f, \varphi_m) \\
&- \int_\Omega 2\mu(T) D(G_0\zeta) : D(\varphi_m) \, dx - \left(G_0 \frac{\partial \zeta}{\partial t}, \varphi_m \right) - b(G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m} + G_0\zeta, \varphi_m) \\
&- b(\widetilde{v}^{\delta}_{\varepsilon m}, G_0\zeta, \varphi_m) \quad \text{a.e. in } (0, \tau).
\end{aligned}$$

We estimate all the terms in the right hand side of the previous equality, we obtain

$$\begin{aligned}
&\left| \left(\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t}, \varphi_m \right) \right| \leq \left(\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^3(\Omega)} \|\nabla \widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{2} \|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^3(\Omega)} \big\| div(\widetilde{v}^{\delta}_{\varepsilon m}) \big\|_{L^2(\Omega)} \right) \|\varphi_m\|_{\mathbf{L}^6(\Omega)} \\
&+ \frac{1}{\delta} \big\| div(\widetilde{v}^{\delta}_{\varepsilon m}) \big\|_{L^2(\Omega)} \big\| div(\varphi_m) \big\|_{L^2(\Omega)} + \mu_* \|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{H}^1(\Omega)} \|\varphi_m\|_{\mathbf{H}^1(\Omega)} \\
&+ \|\ell\|_{\mathbf{L}^2(\Gamma_0)} \|\varphi_m\|_{\mathbf{L}^2(\Gamma_0)} + \|f\|_{\mathbf{L}^2(\Omega)} \|\varphi_m\|_{\mathbf{L}^2(\Omega)} + \mu_* |\zeta| \|G_0\|_{\mathbf{H}^1(\Omega)} \|\varphi_m\|_{\mathbf{H}^1(\Omega)} \\
&+ \left| \frac{\partial \zeta}{\partial t} \right| \|G_0\|_{\mathbf{L}^2(\Omega)} \|\varphi_m\|_{\mathbf{L}^2(\Omega)} + |\zeta| \|G_0\|_{\mathbf{L}^4(\Omega)} \|\nabla \widetilde{v}^{\delta}_{\varepsilon m} + \nabla G_0\|_{\mathbf{L}^2(\Omega)} \|\varphi_m\|_{\mathbf{L}^4(\Omega)} \\
&+ |\zeta| \|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^4(\Omega)} \|\nabla G_0\|_{\mathbf{L}^2(\Omega)} \|\varphi_m\|_{\mathbf{L}^4(\Omega)} \quad \text{a.e. in } (0, \tau).
\end{aligned}$$

By using the classical inequality

$$\|u\|_{L^3(\Omega)} \leq \|u\|^{\frac{1}{2}}_{L^2(\Omega)} \|u\|^{\frac{1}{2}}_{L^6(\Omega)} \quad \forall u \in L^6(\Omega) \cap L^2(\Omega)$$

and the injection of $\mathbf{H}^1(\Omega)$ in $\mathbf{L}^6(\Omega)$, we infer that there exists a constant c, independent of m, δ and ε, such that

$$\|\widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^3(\Omega)} \|\nabla \, \widetilde{v}^{\delta}_{\varepsilon m}\|_{\mathbf{L}^2(\Omega)} \|\varphi_m\|_{\mathbf{L}^6(\Omega)} \leq c \|\widetilde{v}^{\delta}_{\varepsilon m}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)} \|\widetilde{v}^{\delta}_{\varepsilon m}\|^{\frac{3}{2}}_{\mathbf{H}^1(\Omega)} \|\varphi_m\|_{\mathbf{H}^1(\Omega)}.$$

As $(w_j)_{j\geq 1}$ is an orthogonal family of $\mathbf{L}^2(\Omega)$ and φ_m is the orthogonal projection with respect to the inner product of $\mathbf{H}^1(\Omega)$ of φ on $\mathrm{Span}\{w_1,\dots,w_m\}$, we have $\|\varphi_m\|_{\mathbf{H}^1(\Omega)} \leq \|\varphi\|_{\mathbf{H}^1(\Omega)}$ and

$$\left(\frac{\widetilde{v}^\delta_{\varepsilon m}}{\partial t},\varphi_m\right) = \left(\frac{\partial\widetilde{v}^\delta_{\varepsilon m}}{\partial t},\varphi_k\right) \quad \forall k\geq m.$$

Since $(w_j)_{j\geq 1}$ is an Hilbertian basis of $\mathcal{V}_0$, the sequence $(\varphi_k)_{k\geq 1}$ converges strongly to φ in $\mathbf{H}^1(\Omega)$ and we get

$$\left(\frac{\widetilde{v}^\delta_{\varepsilon m}}{\partial t},\varphi_m\right) = \left(\frac{\partial\widetilde{v}^\delta_{\varepsilon m}}{\partial t},\varphi\right).$$

Then, we obtain

$$\begin{aligned}
&\left|\left(\frac{\partial\widetilde{v}^\delta_{\varepsilon m}}{\partial t},\varphi\right)\right| \leq \left(\frac{\sqrt{3}}{2}+1\right) c\|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)}\|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{3}{2}}_{\mathbf{H}^1(\Omega)}\|\varphi\|_{\mathbf{H}^1(\Omega)}\\
&+\frac{\sqrt{3}}{\delta}\big\|div(\widetilde{v}^\delta_{\varepsilon m})\big\|_{L^2(\Omega)}\|\varphi\|_{\mathbf{H}^1(\Omega)} + \mu_*\|\widetilde{v}^\delta_{\varepsilon m}\|_{\mathbf{H}^1(\Omega)}\|\varphi\|_{\mathbf{H}^1(\Omega)} + \tilde{c}\|\ell\|_{\mathbf{L}^2(\Gamma_0)}\|\varphi\|_{\mathbf{H}^1(\Omega)}\\
&+\left(\|f\|_{\mathbf{L}^2(\Omega)} + \mu_*|\zeta|\|G_0\|_{\mathbf{H}^1(\Omega)} + \left|\frac{\partial\zeta}{\partial t}\right|\|G_0\|_{\mathbf{L}^2(\Omega)}\right)\|\varphi\|_{\mathbf{H}^1(\Omega)}\\
&+\left(K^2|\zeta|\|G_0\|_{\mathbf{H}^1(\Omega)}\|\widetilde{v}^\delta_{\varepsilon m}+G_0\zeta\|_{\mathbf{H}^1(\Omega)} + K^2|\zeta|\|\widetilde{v}^\delta_{\varepsilon m}\|_{\mathbf{H}^1(\Omega)}\|G_0\|_{\mathbf{H}^1(\Omega)}\right)\|\varphi\|_{\mathbf{H}^1(\Omega)} \text{ a.e. in } (0,\tau),
\end{aligned}$$

where $\tilde{c}$ is the norm of the trace operator $\gamma_0 : \mathbf{H}^1(\Omega) \to \mathbf{L}^2(\Gamma_0)$. Hence

$$\begin{aligned}
&\left\|\frac{\partial\widetilde{v}^\delta_{\varepsilon m}}{\partial t}\right\|_{\mathcal{V}_0'} \leq \left(\frac{\sqrt{3}}{2}+1\right) c\|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)}\|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{3}{2}}_{\mathbf{H}^1(\Omega)} + \frac{\sqrt{3}}{\delta}\big\|div(\widetilde{v}^\delta_{\varepsilon m})\big\|_{L^2(\Omega)} + \mu_*\|\widetilde{v}^\delta_{\varepsilon m}\|_{\mathbf{H}^1(\Omega)}\\
&+\tilde{c}\|\ell\|_{\mathbf{L}^2(\Gamma_0)} + \|f\|_{\mathbf{L}^2(\Omega)} + \mu_*|\zeta|\|G_0\|_{\mathbf{H}^1(\Omega)} + \left|\frac{\partial\zeta}{\partial t}\right|\|G_0\|_{\mathbf{L}^2(\Omega)}\\
&+K^2|\zeta|\|G_0\|_{\mathbf{H}^1(\Omega)}\|\widetilde{v}^\delta_{\varepsilon m}+G_0\zeta\|_{\mathbf{H}^1(\Omega)} + K^2|\zeta|\|\widetilde{v}^\delta_{\varepsilon m}\|_{\mathbf{H}^1(\Omega)}\|G_0\|_{\mathbf{H}^1(\Omega)} \quad \text{a.e. in } (0,\tau).
\end{aligned}$$

Observing that

$$\begin{aligned}
\int_0^\tau \left[\|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{1}{2}}_{\mathbf{L}^2(\Omega)}\|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{3}{2}}_{\mathbf{H}^1(\Omega)}\right]^{\frac{4}{3}} dt &= \int_0^\tau \|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{2}{3}}_{\mathbf{L}^2(\Omega)}\|\widetilde{v}^\delta_{\varepsilon m}\|^2_{\mathbf{H}^1(\Omega)}\,dt\\
&\leq \|\widetilde{v}^\delta_{\varepsilon m}\|^{\frac{2}{3}}_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}\|\widetilde{v}^\delta_{\varepsilon m}\|^2_{L^2(0,\tau;\mathbf{H}^1(\Omega))},
\end{aligned}$$

we infer from the estimates of Lemma 4.1 that there exists a constant $C_\delta > 0$, independent of m and ε, such that

$$\int_0^\tau \left\|\frac{\partial\widetilde{v}^\delta_{\varepsilon m}}{\partial t}\right\|^{\frac{4}{3}}_{\mathcal{V}_0'} dt \leq C_\delta$$

which concludes the proof. □ □

In order to pass to the limit as m tends to $+\infty$, we will use also the following Lemma.

Lemma 4.3. *Let $\varepsilon > 0$ and $\ell \in L^2\big(0,\tau;\mathbf{L}^2_+(\Gamma_0)\big) \cap L^\infty\big(0,\tau;\mathbf{L}^\infty_+(\Gamma_0)\big)$. Then the mapping Ψ'_ε is Lipschitz continuous from $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$ to $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$.*

Proof. Let us recall that, for all $u \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$, $\Psi'_\varepsilon(u) \in \big(L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)\big)' = L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$ is defined by

$$\langle \Psi'_\varepsilon(u), w\rangle = \int_0^\tau \int_{\Gamma_0} \ell \frac{u\cdot w}{\sqrt{\varepsilon^2+|u|^2}}\, dx'dt \quad \forall w \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$, i.e.

$$\Psi'_\varepsilon(u) = \ell \frac{u}{\sqrt{\varepsilon^2+|u|^2}} \quad \forall u \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big).$$

But the mapping

$$h_\varepsilon : \begin{cases} \mathbb{R}^d \to \mathbb{R}^d \\ u \mapsto \dfrac{u}{\sqrt{\varepsilon^2+|u|^2}} \end{cases}$$

is Fréchet differentiable on $\mathbb{R}^d$ and

$$\mathrm{Jac}(h_\varepsilon)(u) = \left(\frac{\partial h_{\varepsilon i}}{\partial x_j}(u)\right)_{1\le i,j\le d} = \left(\frac{\delta_{i,j}}{\sqrt{\varepsilon^2+|u|^2}} - \frac{u_i u_j}{\big(\varepsilon^2+|u|^2\big)^{\frac{3}{2}}}\right)_{1\le i,j\le d}$$

where $\delta_{i,j}=1$ if $i=j$ and $\delta_{i,j}=0$ if $i\neq j$. It follows that

$$\left|\frac{\partial h_{\varepsilon i}}{\partial x_j}(u)\right| \le \frac{2}{\varepsilon} \quad \forall i,j \in \{1,\dots,d\},\ \forall u \in \mathbb{R}^d$$

and h_ε is Lipschitz continuous on $\mathbb{R}^d$. Since $\ell \in L^\infty\big(0,\tau;\mathbf{L}^\infty_+(\Gamma_0)\big)$, we infer that Ψ'_ε is Lipschitz continuous from $L^2\big(0,\tau;\mathbf{L}^2_+(\Gamma_0)\big)$ into $L^2\big(0,\tau;\mathbf{L}^2_+(\Gamma_0)\big)$. □ □

Now, by using the estimates obtained in Lemma 4.1 and Lemma 4.2 combined with compactness arguments, we can prove the following existence result for the penalized problems (P^δ_ε).

Theorem 4.4. *Let $\varepsilon > 0$ and $\delta > 0$. Assume that (8), (9) and (10) hold and that $(\widetilde{v}^\delta_{\varepsilon 0})_{\varepsilon>0,\delta>0}$ is a bounded sequence of $\mathbf{L}^2(\Omega)$. Then, there exists a subsequence of $(\widetilde{v}^\delta_{\varepsilon m})_{m\ge 1}$, still denoted $(\widetilde{v}^\delta_{\varepsilon m})_{m\ge 1}$, such that*

$$\widetilde{v}^\delta_{\varepsilon m} \rightharpoonup \widetilde{v}^\delta_\varepsilon \quad \textit{weakly star in } L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big) \tag{32}$$

$$\widetilde{v}^\delta_{\varepsilon m} \rightharpoonup \widetilde{v}^\delta_\varepsilon \quad \textit{weakly in } L^2(0,\tau;\mathcal{V}_0) \tag{33}$$

and $\widetilde{v}^\delta_\varepsilon$ is solution of (P^δ_ε). Furthermore $\dfrac{\partial \widetilde{v}^\delta_\varepsilon}{\partial t}$ belongs to $L^{\frac{4}{3}}(0,\tau;\mathcal{V}'_0)$.

Proof. The convergences (32)-(33) follow immediately from the a priori estimates (25)-(26) obtained in Lemma 4.1. From the estimate (31) obtained in Lemma 4.2, we infer that, possibly extracting another subsequence still denoted denoted $(\widetilde{v}^{\delta}_{\varepsilon m})_{m\geq 1}$, we have

$$\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t} \rightharpoonup \frac{\partial \widetilde{v}^{\delta}_{\varepsilon}}{\partial t} \quad \text{weakly in } L^{\frac{4}{3}}(0,\tau;\mathcal{V}_0'). \tag{34}$$

By using Aubin's lemma [22] and the convergences (33) and (34), with $X_0 = \mathcal{V}_0$, $X = \mathbf{L}^4(\Omega)$ and $X_1 = \mathcal{V}_0'$ we obtain

$$\widetilde{v}^{\delta}_{\varepsilon m} \to \widetilde{v}^{\delta}_{\varepsilon} \quad \text{strongly in } L^2(0,\tau;\mathbf{L}^4(\Omega)).$$

We may use again Aubin's lemma with $X_0 = \mathcal{V}_0$, $X = \mathbf{H}^s(\Omega)$ and $X_1 = \mathcal{V}_0'$ with $\frac{1}{2} < s < 1$: the embedding of X_0 into X is compact, so we obtain

$$\widetilde{v}^{\delta}_{\varepsilon m} \to \widetilde{v}^{\delta}_{\varepsilon} \quad \text{strongly in } L^2(0,\tau;\mathbf{H}^s(\Omega)).$$

Then, with trace theorem [16], we infer that

$$\widetilde{v}^{\delta}_{\varepsilon m} \to \widetilde{v}^{\delta}_{\varepsilon} \quad \text{strongly in } L^2(0,\tau;\mathbf{L}^2(\Gamma_0))$$

where we identify here the functions $\widetilde{v}^{\delta}_{\varepsilon m}$ and $\widetilde{v}^{\delta}_{\varepsilon}$ with their trace on Γ_0.

Now, using (32)-(34) and Simon's lemma [20] and possibly extracting another subsequence, still denoted $(\widetilde{v}^{\delta}_{\varepsilon m})_{m\geq 1}$, we obtain

$$\widetilde{v}^{\delta}_{\varepsilon m} \to \widetilde{v}^{\delta}_{\varepsilon} \quad \text{strongly in } \mathcal{C}^0(0,\tau;H), \tag{35}$$

for any Banach space H such that $\mathbf{L}^2(\Omega) \subset H \subset \mathcal{V}_0'$ with continuous injections and compact embedding of $\mathbf{L}^2(\Omega)$ into H.

Let $\chi \in \mathcal{D}(0,\tau)$ and $\varphi \in \mathcal{V}_0$. For all $m \geq 1$ we define again φ_m as the othogonal projection with respect to the inner product of $\mathbf{H}^1(\Omega)$ of φ on Span$\{w_1,\dots,w_m\}$. With (22) we have

$$\begin{aligned}
&\int_0^\tau \left[\left(\frac{\partial \widetilde{v}^{\delta}_{\varepsilon m}}{\partial t}, \varphi_m\right) + b(\widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}, \varphi_m) + \frac{1}{2}\int_\Omega \widetilde{v}^{\delta}_{\varepsilon m} div(\widetilde{v}^{\delta}_{\varepsilon m})\varphi_m\, dx\right]\chi\, dt \\
&+\frac{1}{\delta}\int_0^\tau \Big(div(\widetilde{v}^{\delta}_{\varepsilon m}), div(\varphi_m)\chi\Big)\, dt + a(T;\widetilde{v}^{\delta}_{\varepsilon m}, \varphi_m\chi) + \Big\langle \Psi'_\varepsilon(\widetilde{v}^{\delta}_{\varepsilon m}), \varphi_m\chi\Big\rangle = \int_0^\tau (f,\varphi_m)\chi\, dt \\
&-a(T;G_0\zeta,\varphi_m\chi) - \int_0^\tau \left[\left(G_0\frac{\partial \zeta}{\partial t}, \varphi_m\right) + b(G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m} + G_0\zeta, \varphi_m) + b(\widetilde{v}^{\delta}_{\varepsilon m}, G_0\zeta, \varphi_m)\right]\chi\, dt.
\end{aligned}$$

With an integration by parts of the first term we get

$$\begin{aligned}
&\int_0^\tau \Big(\widetilde{v}^{\delta}_{\varepsilon m}, \varphi_m\Big)\frac{\partial \chi}{\partial t}\, dt + \int_0^\tau \left[b(\widetilde{v}^{\delta}_{\varepsilon m}, \widetilde{v}^{\delta}_{\varepsilon m}, \varphi_m) + \frac{1}{2}\int_\Omega \widetilde{v}^{\delta}_{\varepsilon m} div(\widetilde{v}^{\delta}_{\varepsilon m})\varphi_m\, dx\right]\chi\, dt \\
&+\frac{1}{\delta}\int_0^\tau \Big(div(\widetilde{v}^{\delta}_{\varepsilon m}), div(\varphi_m)\chi\Big)\, dt + a(T;\widetilde{v}^{\delta}_{\varepsilon m}, \varphi_m\chi) + \Big\langle \Psi'_\varepsilon(\widetilde{v}^{\delta}_{\varepsilon m}), \varphi_m\chi\Big\rangle = \int_0^\tau (f,\varphi_m)\chi\, dt \\
&-a(T;G_0\zeta,\varphi_m\chi) - \int_0^\tau \left[\left(G_0\frac{\partial \zeta}{\partial t}, \varphi_m\right) + b(G_0\zeta, \widetilde{v}^{\delta}_{\varepsilon m} + G_0\zeta, \varphi_m) + b(\widetilde{v}^{\delta}_{\varepsilon m}, G_0\zeta, \varphi_m)\right]\chi\, dt.
\end{aligned}$$

Reminding that $(\varphi_m)_{m\geq 1}$ converges strongly to φ in $\mathbf{H}^1(\Omega)$ and using Lemma 4.3, we can pass to the limit in all the terms and we obtain

$$\int_0^\tau \left(\widetilde{v}_\varepsilon^\delta,\varphi\right)\frac{\partial\chi}{\partial t}\,dt+\int_0^\tau\left[b(\widetilde{v}_\varepsilon^\delta,\widetilde{v}_\varepsilon^\delta,\varphi)+\frac{1}{2}\int_\Omega \widetilde{v}_\varepsilon^\delta div(\widetilde{v}_\varepsilon^\delta)\varphi\,dx+\frac{1}{\delta}\left(div(\widetilde{v}_\varepsilon^\delta),div(\varphi)\right)\right]\chi\,dt$$
$$+a(T;\widetilde{v}_\varepsilon^\delta,\varphi\chi)+\left\langle\Psi_\varepsilon'(\widetilde{v}_\varepsilon^\delta),\varphi\chi\right\rangle=\int_0^\tau (f,\varphi)\chi\,dt-a(T;G_0\zeta,\varphi\chi)$$
$$-\int_0^\tau\left[\left(G_0\frac{\partial\zeta}{\partial t},\varphi\right)+b(G_0\zeta,\widetilde{v}_\varepsilon^\delta+G_0\zeta,\varphi)+b(\widetilde{v}_\varepsilon^\delta,G_0\zeta,\varphi)\right]\chi\,dt.$$

which gives (18). It remains to chek that the initial condition (19) is satisfied. Indeed, with (35), we have

$$\widetilde{v}_{\varepsilon m}^\delta(0)\to\widetilde{v}_\varepsilon^\delta(0)\quad\text{strongly in } H$$

with $\mathbf{L}^2(\Omega)\subset H\subset\mathcal{V}_0'$ and we have also

$$\widetilde{v}_{\varepsilon m}^\delta(0)=\widetilde{v}_{\varepsilon m0}^\delta\to\widetilde{v}_{\varepsilon 0}^\delta\quad\text{strongly in } \mathbf{L}^2(\Omega).$$

Hence $\widetilde{v}_\varepsilon^\delta(0)=\widetilde{v}_{\varepsilon 0}^\delta$. □ □

5 Properties of the approximate pressure

For any $\varepsilon>0$ and $\delta>0$ we define and approximate pressure $p_\varepsilon^\delta\in L^2\big(0,\tau;L^2(\Omega)\big)$ by

$$p_\varepsilon^\delta=-\frac{1}{\delta}div(\widetilde{v}_\varepsilon^\delta)\tag{36}$$

where $\widetilde{v}_\varepsilon^\delta$ is the solution of the penalized problem (P_ε^δ) obtained in the previous Section. From (18) we get

$$\left\langle\frac{d}{dt}\left(\widetilde{v}_\varepsilon^\delta,\varphi\right),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}+\left\langle b(\widetilde{v}_\varepsilon^\delta,\widetilde{v}_\varepsilon^\delta,\varphi),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}$$
$$+\frac{1}{2}\langle\int_\Omega\widetilde{v}_\varepsilon^\delta div(\widetilde{v}_\varepsilon^\delta)\varphi\,dx,\chi\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}-\left\langle\left(p_\varepsilon^\delta,div(\varphi)\right),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}+a(T;\widetilde{v}_\varepsilon^\delta,\varphi\chi)$$
$$+\left\langle\Psi_\varepsilon'(\widetilde{v}_\varepsilon^\delta),\varphi\chi\right\rangle=\left\langle(f,\varphi),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}-a(T;G_0\zeta,\varphi\chi)$$
$$-\left\langle\left(G_0\frac{\partial\zeta}{\partial t},\varphi\right),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}-\left\langle b(G_0\zeta,\widetilde{v}_\varepsilon^\delta+G_0\zeta,\varphi),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}$$
$$-\left\langle b(\widetilde{v}_\varepsilon^\delta,G_0\zeta,\varphi),\chi\right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)},\quad\forall\varphi\in\mathcal{V}_0,\ \forall\chi\in\mathcal{D}(0,\tau).\tag{37}$$

Furthermore, with Green's formula, we obtain

$$\int_\Omega p_\varepsilon^\delta\,dx=-\frac{1}{\delta}\int_{\partial\Omega}\widetilde{v}_\varepsilon^\delta\cdot n\,dx=0\quad\text{a.e. in } (0,\tau)\tag{38}$$

and, with (27) and (33), we have

$$\left\|p_\varepsilon^\delta\right\|_{L^2(0,\tau;L^2(\Omega))}\leq\frac{C}{\sqrt{\delta}}$$

where C is a constant independent of δ and ε. Unfortunately this last estimate does not allow us to pass to the limit in the term $\langle (p_\varepsilon^\delta, div(\varphi)), \chi \rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}$ as δ tends to zero. So we will establish an estimate independent of ε and δ by using the same kind of technique as in [3]).

Lemma 5.1. *Under the same assumptions as in Lemma 4.1, there exists a constant C, independent of δ and ε, such that*

$$\|p_\varepsilon^\delta\|_{H^{-1}(0,\tau;L^2(\Omega))} \leq C. \tag{39}$$

Proof. Let $\chi \in \mathcal{D}(0,\tau)$ and $w \in L_0^2(\Omega)$. Then there exists $\varphi \in \mathbf{H}_0^1(\Omega)$ such that

$$div(\varphi) = w \quad \text{in } \Omega$$

and $\varphi = P(w)$ where P is a linear continuous operator from $L_0^2(\Omega)$ into $\mathbf{H}_0^1(\Omega)$ (see [15]). With an integration by parts of the first term of (37), we get

$$\int_0^\tau (p_\varepsilon^\delta, div(\varphi))\chi\, dt = -\int_0^\tau \left(\widetilde{v}_\varepsilon^\delta, \varphi\right)\frac{\partial \chi}{\partial t}\, dt + \int_0^\tau \left[b(\widetilde{v}_\varepsilon^\delta, \widetilde{v}_\varepsilon^\delta, \varphi) + \frac{1}{2}\int_\Omega \widetilde{v}_\varepsilon^\delta div(\widetilde{v}_\varepsilon^\delta)\varphi\, dx\right]\chi\, dt$$
$$+a(T; \widetilde{v}_\varepsilon^\delta, \varphi\chi) - \int_0^\tau (f,\varphi)\chi\, dt + a(T; G_0\zeta, \varphi\chi)$$
$$+\int_0^\tau \left[\left(G_0\frac{\partial \zeta}{\partial t}, \varphi\right) + b(G_0\zeta, \widetilde{v}_\varepsilon^\delta + G_0\zeta, \varphi) + b(\widetilde{v}_\varepsilon^\delta, G_0\zeta, \varphi)\right]\chi\, dt$$

Let us denote $\varphi\chi = \eta$ and recall that K is the constant of the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$. We get

$$\left|\int_0^\tau \left[b(\widetilde{v}_\varepsilon^\delta, \widetilde{v}_\varepsilon^\delta, \varphi) + \frac{1}{2}\int_\Omega \widetilde{v}_\varepsilon^\delta div(\widetilde{v}_\varepsilon^\delta)\varphi\, dx\right]\chi\, dt\right| \leq$$
$$\leq \left(1 + \frac{\sqrt{3}}{2}\right)\|\widetilde{v}_\varepsilon^\delta\|_{L^2(0,\tau;\mathbf{L}^4(\Omega))}\|\widetilde{v}_\varepsilon^\delta\|_{L^2(0,\tau;\mathbf{H}^1(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{L}^4(\Omega))}$$
$$\leq K^2\left(1 + \frac{\sqrt{3}}{2}\right)\|\widetilde{v}_\varepsilon^\delta\|^2_{L^2(0,\tau;\mathbf{H}^1(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))}.$$

Then we obtain

$$\left|\int_0^\tau (p_\varepsilon^\delta, w)\chi\, dt\right| \leq \|\widetilde{v}_\varepsilon^\delta\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))}\|\frac{\partial \eta}{\partial t}\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))}$$
$$+K^2\left(1 + \frac{\sqrt{3}}{2}\right)\|\widetilde{v}_\varepsilon^\delta\|^2_{L^2(0,\tau;\mathbf{H}^1(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))}$$
$$+\mu_*\sqrt{\tau}\|\widetilde{v}_\varepsilon^\delta\|_{L^2(0,\tau;\mathbf{H}^1(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))} + \sqrt{\tau}\|f\|_{L^2(0,\tau;\mathbf{L}^2(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}$$
$$+\mu_*\|G_0\|_{\mathbf{H}^1(\Omega)}\|\zeta\|_{L^1(0,\tau)}\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))} + \|G_0\|_{\mathbf{L}^2(\Omega)}\left\|\frac{\partial \zeta}{\partial t}\right\|_{L^1(0,\tau)}\|\eta\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}$$
$$+K^2\|G_0\|_{\mathbf{H}^1(\Omega)}\|\zeta\|_{L^2(0,\tau)}\|\widetilde{v}_\varepsilon^\delta + G_0\zeta\|_{L^2(0,\tau;\mathbf{H}^1(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))}$$
$$+K^2\|G_0\|_{\mathbf{H}^1(\Omega)}\|\zeta\|_{L^2(0,\tau)}\|\widetilde{v}_\varepsilon^\delta\|_{L^2(0,\tau;\mathbf{H}^1(\Omega))}\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))}.$$

By using the continuity of the operator P and the continuous injection of $H^1(0,\tau)$ into $L^\infty(0,\tau)$, we have

$$\|\eta\|_{L^\infty(0,\tau;\mathbf{H}^1(\Omega))} = \|\chi\|_{L^\infty(0,\tau)}\big\|P(w)\big\|_{\mathbf{H}^1(\Omega)} \leq C\|\chi\|_{H^1(0,\tau)}\|w\|_{\mathbf{L}^2(\Omega)} \leq C\|\eta\|_{H^1(0,\tau;\mathbf{L}^2(\Omega))},$$

where C is a constant independent of δ and ε. With the estimates (25) and (26), we infer that $\widetilde{v}_\varepsilon^\delta$ is bounded in $L^2(0,\tau;\mathbf{H}^1(\Omega))\cap L^\infty(0,\tau;\mathbf{L}^2(\Omega))$ independently of δ and ε, and we obtain

$$\left|\int_0^\tau (p_\varepsilon^\delta, w)\chi\, dt\right| \le C\|w\chi\|_{H^1(0,\tau;L^2(\Omega))} \quad \forall w\in L_0^2(\Omega),\ \forall\chi\in\mathcal{D}(0,\tau) \tag{40}$$

where we denote again by C a constant independent of δ and ε.

Let now $\widetilde{w}\in L^2(\Omega)$. We can apply (40) with

$$w=\widetilde{w}-\frac{1}{mes\Omega}\int_\Omega \widetilde{w}\, dx.$$

Indeed, $w\in L_0^2(\Omega)$. Furthermore, with (38), we have

$$\left|\int_0^\tau \left(p_\varepsilon^\delta, \widetilde{w}-\frac{1}{mes\Omega}\int_\Omega \widetilde{w}\, dx\right)\chi\, dt\right| =$$
$$=\left|\int_0^\tau (p_\varepsilon^\delta,\widetilde{w})\chi\, dt-\frac{1}{mes\Omega}\int_0^\tau\left(\int_\Omega p_\varepsilon^\delta\, dx\right)\left(\int_\Omega \widetilde{w}\, dx\right)\chi\, dt\right| = \left|\int_0^\tau (p_\varepsilon^\delta,\widetilde{w})\chi\, dt\right|.$$

Observing that $\|w\|_{L^2(\Omega)}\le\|\widetilde{w}\|_{L^2(\Omega)}$, we obtain

$$\left|\int_0^\tau (p_\varepsilon^\delta, w)\chi\, dt\right| = \left|\int_0^\tau (p_\varepsilon^\delta,\widetilde{w})\chi\, dt\right| \le C\|\widetilde{w}\chi\|_{H^1(0,\tau;L^2(\Omega))},\ \forall\widetilde{w}\in L^2(\Omega),\ \forall\chi\in\mathcal{D}(0,\tau). \tag{41}$$

Then the density of $\mathcal{D}(0,\tau)\otimes L^2(\Omega)$ into $H_0^1(0,\tau;L^2(\Omega))$ allows us to conclude. □ □

6 Existence results for the problems (P_ε) and (P)

Now we can pass to the limit in the penalized problems (P_ε^δ) when δ tends to zero.

Theorem 6.1. *Let $\varepsilon>0$ and assume that $(\widetilde{v}_{\varepsilon 0}^\delta)_{\varepsilon>0,\delta>0}$ is a bounded sequence of $\mathbf{L}^2(\Omega)$. Assume moreover that (8), (9), (10) and (20) hold. Then, there exists a subsequence of $(\widetilde{v}_\varepsilon^\delta, p_\varepsilon^\delta)_{\delta>0}$, still denoted $(\widetilde{v}_\varepsilon^\delta, p_\varepsilon^\delta)_{\delta>0}$, such that*

$$\widetilde{v}_\varepsilon^\delta \rightharpoonup \widetilde{v}_\varepsilon \quad \text{weakly star in } L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big) \tag{42}$$

$$\widetilde{v}_\varepsilon^\delta \rightharpoonup \widetilde{v}_\varepsilon \quad \text{weakly in } L^2(0,\tau;\mathcal{V}_0) \tag{43}$$

$$\widetilde{p}_\varepsilon^\delta \rightharpoonup \widetilde{p}_\varepsilon \quad \text{weakly in } H^{-1}\big(0,\tau;L_0^2(\Omega)\big) \tag{44}$$

and $(\widetilde{v}_\varepsilon, p_\varepsilon)$ is solution of (P_ε). Furthermore $\dfrac{\partial\widetilde{v}_\varepsilon}{\partial t}$ belongs to $L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big)$.

Proof. Observing that the estimates obtained in Lemma 4.1 are independent of m, δ and ε, we infer that the sequence $(\widetilde{v}_\varepsilon^\delta, p_\varepsilon^\delta)_{\delta>0}$ is bounded in $L^2(0,\tau;\mathcal{V}_0)\cap L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big)$. Moreover, with Proposition 5.1, the sequence $(p_\varepsilon^\delta)_{\delta>0}$ is bounded in $H^{-1}\big(0,\tau;\mathbf{L}^2(\Omega)\big)$ and the convergences (43)-(42)-(44) follow immediately.

From (27) we infer that

$$\left\| div(\widetilde{v}_\varepsilon^\delta) \right\|_{L^2(0,\tau;L^2(\Omega))} \le C\sqrt{\delta}$$

with a constant C independent of δ and ε. Thus

$$div(\widetilde{v}_\varepsilon^\delta) \to 0 \quad \text{strongly in } L^2\big(0,\tau;L^2(\Omega)\big).$$

Finally we can obtain an estimate of $\dfrac{\partial \widetilde{v}_\varepsilon^\delta}{\partial t}$ in $L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big)$ by using the same kind of computations as in Lemma 4.2. Indeed, let $\varphi \in \mathcal{V}_{0div}$ and $\chi \in \mathcal{D}(0,\tau)$. With (37) we get

$$\begin{aligned}
&\left\langle \frac{d}{dt}\left(\widetilde{v}_\varepsilon^\delta, \varphi\right), \chi \right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} = \int_0^\tau \int_\Omega \frac{\partial \widetilde{v}_\varepsilon^\delta}{\partial t} \varphi\chi \, dxdt = -\big\langle b(\widetilde{v}_\varepsilon^\delta, \widetilde{v}_\varepsilon^\delta, \varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} \\
&-\frac{1}{2}\Big\langle \int_\Omega \widetilde{v}_\varepsilon^\delta div(\widetilde{v}_\varepsilon^\delta)\varphi \, dx, \chi \Big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - a(T; \widetilde{v}_\varepsilon^\delta, \varphi\chi) - \big\langle \Psi'_\varepsilon(\widetilde{v}_\varepsilon^\delta), \varphi\chi \big\rangle \\
&+\big\langle (f,\varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - a(T; G_0\zeta, \varphi\chi) - \left\langle \left(G_0 \frac{\partial \zeta}{\partial t}, \varphi \right), \chi \right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} \\
&-\big\langle b(G_0\zeta, \widetilde{v}_\varepsilon^\delta + G_0\zeta, \varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - \big\langle b(\widetilde{v}_\varepsilon^\delta, G_0\zeta, \varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}.
\end{aligned}$$

We can estimate all the terms in the right hand side of the previous equality and we obtain

$$\begin{aligned}
&\left| \int_0^\tau \int_\Omega \frac{\partial \widetilde{v}_\varepsilon^\delta}{\partial t} \varphi\chi \, dxdt \right| \le \left(\frac{\sqrt{3}}{2} + 1 \right) c \int_0^\tau \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{H}^1(\Omega)}^{\frac{3}{2}} \|\varphi\|_{\mathbf{H}^1(\Omega)} |\chi| \, dt \\
&+\mu_* \int_0^\tau \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{H}^1(\Omega)} \|\varphi\|_{\mathbf{H}^1(\Omega)} |\chi| \, dt + \tilde{c} \int_0^\tau \|\ell\|_{\mathbf{L}^2(\Gamma_0)} \|\varphi\|_{\mathbf{H}^1(\Omega)} |\chi| \, dt \\
&+ \int_0^\tau \left[\|f\|_{\mathbf{L}^2(\Omega)} + \mu_* |\zeta| \|G_0\|_{\mathbf{H}^1(\Omega)} + \left| \frac{\partial \zeta}{\partial t} \right| \|G_0\|_{\mathbf{L}^2(\Omega)} \right] \|\varphi\|_{\mathbf{H}^1(\Omega)} |\chi| \, dt \\
&+ \int_0^\tau \left[K^2 |\zeta| \|G_0\|_{\mathbf{H}^1(\Omega)} \|\widetilde{v}_\varepsilon^\delta + G_0\zeta\|_{\mathbf{H}^1(\Omega)} + K^2 |\zeta| \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{H}^1(\Omega)} \|G_0\|_{\mathbf{H}^1(\Omega)} \right] \|\varphi\|_{\mathbf{H}^1(\Omega)} |\chi| \, dt.
\end{aligned}$$

Observing that

$$\begin{aligned}
&\int_0^\tau \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{L}^2(\Omega)}^{\frac{1}{2}} \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{H}^1(\Omega)}^{\frac{3}{2}} \|\varphi\|_{\mathbf{H}^1(\Omega)} |\chi| \, dt \le \\
&\le \|\widetilde{v}_\varepsilon^\delta\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}^{\frac{1}{2}} \left(\int_0^\tau \|\widetilde{v}_\varepsilon^\delta\|_{\mathbf{H}^1(\Omega)}^2 \, dt \right)^{\frac{3}{4}} \|\varphi\|_{\mathbf{H}^1(\Omega)} \|\chi\|_{L^4(0,\tau)} \\
&\le \|\widetilde{v}_\varepsilon^\delta\|_{L^\infty(0,\tau;\mathbf{L}^2(\Omega))}^{\frac{1}{2}} \|\widetilde{v}_\varepsilon^\delta\|_{L^2(0,\tau;\mathbf{H}^1(\Omega))}^{\frac{3}{2}} \|\varphi\chi\|_{L^4(0,\tau;\mathbf{H}^1(\Omega))}
\end{aligned}$$

and reminding that $(\widetilde{v}_\varepsilon^\delta)_{\delta>0}$ is bounded in $L^2\big(0,\tau;\mathbf{H}^1(\Omega)\big) \cap L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big)$ independently of δ and ε, we infer that

$$\left\| \frac{\partial \widetilde{v}_\varepsilon^\delta}{\partial t} \right\|_{L^{\frac{4}{3}}(0,\tau;(\mathcal{V}_{0div})')} \le C \tag{45}$$

with a constant C independent of δ and ε.

It follows that, possibly extracting another subsequence still denoted $(\widetilde{v}_\varepsilon^\delta)_{\delta>0}$, we have

$$\frac{\partial \widetilde{v}_\varepsilon^\delta}{\partial t} \rightharpoonup \frac{\partial \widetilde{v}_\varepsilon}{\partial t} \quad \text{weakly in } L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big). \tag{46}$$

By using Aubin's lemma, with $X_0 = \mathcal{V}_0$, $X = \mathbf{L}^4(\Omega)$ and $X_1 = (\mathcal{V}_{0div})'$, we obtain

$$\widetilde{v}_\varepsilon^\delta \to \widetilde{v}_\varepsilon \quad \text{strongly in } L^2\big(0,\tau;\mathbf{L}^4(\Omega)\big)$$

and, with $X_0 = \mathcal{V}_0$, $X = \mathbf{H}^s(\Omega)$, $\dfrac{1}{2} < s < 1$, and $X_1 = (\mathcal{V}_{0div})'$,

$$\widetilde{v}_\varepsilon^\delta \to \widetilde{v}_\varepsilon \quad \text{strongly in } L^2\big(0,\tau;\mathbf{H}^s(\Omega)\big).$$

Hence

$$\widetilde{v}_\varepsilon^\delta \to \widetilde{v}_\varepsilon \quad \text{strongly in } L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big).$$

Finally, using (42)-(46) and Simon's lemma, and possibly extracting another subsequence still denoted $(\widetilde{v}_\varepsilon^\delta)_{\delta>0}$, we obtain

$$\widetilde{v}_\varepsilon^\delta \to \widetilde{v}_\varepsilon \quad \text{strongly in } \mathcal{C}^0(0,\tau;H)$$

for any Banach space H such that $\mathbf{L}^2(\Omega) \subset H \subset (\mathcal{V}_{0div})'$ with continuous injections and compact embedding of $\mathbf{L}^2(\Omega)$ into H.

With all these convergences and with the assumption (20), we can pass to the limit in (18) and (19) by the same techniques as in Theorem 4.4 and we get (16) and (17). □ □

Now, observing that Ψ_ε is convex, we obtain that

$$\Psi_\varepsilon(\widetilde{v}_\varepsilon + \varphi\chi) - \Psi_\varepsilon(\widetilde{v}_\varepsilon) \geq \big\langle \Psi'_\varepsilon(\widetilde{v}_\varepsilon), \varphi\chi \big\rangle \quad \forall \varphi \in \mathcal{V}_0,\ \forall \chi \in \mathcal{D}(0,\tau)$$

and in (16) we get

$$\begin{aligned}
&\left\langle \frac{d}{dt}\left(\widetilde{v}_\varepsilon, \varphi\right), \chi \right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} + \big\langle b(\widetilde{v}_\varepsilon, \widetilde{v}_\varepsilon, \varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - \big\langle \big(p_\varepsilon, div(\varphi)\big), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} \\
&+a(T;\widetilde{v}_\varepsilon, \varphi\chi) + \Psi_\varepsilon(\widetilde{v}_\varepsilon + \varphi\chi) - \Psi_\varepsilon(\widetilde{v}_\varepsilon) \geq \big\langle (f,\varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - a(T; G_0\zeta, \varphi\chi) \\
&-\left\langle \left(G_0 \frac{\partial \zeta}{\partial t}, \varphi \right), \chi \right\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} - \big\langle b(G_0\zeta, \widetilde{v}_\varepsilon + G_0\zeta, \varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)} \\
&-\big\langle b(\widetilde{v}_\varepsilon, G_0\zeta, \varphi), \chi \big\rangle_{\mathcal{D}'(0,\tau),\mathcal{D}(0,\tau)}
\end{aligned} \tag{47}$$

for all $\varphi \in \mathcal{V}_0$ and for all $\chi \in \mathcal{D}(0,\tau)$, with the initial condition

$$\widetilde{v}_\varepsilon(0,\cdot) = \widetilde{v}_{\varepsilon 0}. \tag{48}$$

In order to pass to the limit as ε tends to zero in the previous inequality, we use the following lemma.

Lemma 6.2. *Let $(w_\varepsilon)_{\varepsilon>0}$ be a sequence of $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$ and $w \in L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$ such that $(w_\varepsilon)_{\varepsilon>0}$ converges strongly to w in $L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big)$. Then $\lim_{\varepsilon\to 0}\Psi_\varepsilon(w_\varepsilon) = \Psi(w)$.*

Proof. Let $\varepsilon > 0$. By definition of Ψ_ε and Ψ, we have

$$\Psi_\varepsilon(w_\varepsilon) - \Psi(w) = \int_0^\tau \int_{\Gamma_0} \ell\big(|w_\varepsilon| - |w|\big)\,dx'dt + \int_0^\tau \int_{\Gamma_0} \ell\big(\sqrt{\varepsilon^2 + |w_\varepsilon|^2} - |w_\varepsilon|\big)\,dx'dt.$$

It follows that

$$\begin{aligned}\big|\Psi_\varepsilon(w_\varepsilon) - \Psi(w)\big| &\le \int_0^\tau \int_{\Gamma_0} \ell\big||w_\varepsilon| - |w|\big|\,dx'dt + \int_0^\tau \int_{\Gamma_0} \ell\varepsilon\,dx'dt \\ &\le \|\ell\|_{L^2(0,\tau;\mathbf{L}^2(\Gamma_0))}\big(\|w_\varepsilon - w\|_{L^2(0,\tau;\mathbf{L}^2(\Gamma_0))} + \varepsilon\sqrt{\tau\mathrm{meas}(\Gamma_0)}\big)\end{aligned}$$

which allows us to conclude. □ □

Now we can prove that problem (P) admits a solution.

Theorem 6.3. *Assume that $(\widetilde{v}^\delta_{\varepsilon 0})_{\varepsilon>0,\delta>0}$ is a bounded sequence of $\mathbf{L}^2(\Omega)$. Assume moreover that (8), (9), (10) and (15) hold. Then, there exists a subsequence of $(\widetilde{v}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$, still denoted $(\widetilde{v}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ such that*

$$\widetilde{v}_\varepsilon \rightharpoonup \widetilde{v} \quad \textit{weakly star in } L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big) \tag{49}$$

$$\widetilde{v}_\varepsilon \rightharpoonup \widetilde{v} \quad \textit{weakly in } L^2(0,\tau;\mathcal{V}_0) \tag{50}$$

$$\widetilde{p}_\varepsilon \rightharpoonup \widetilde{p} \quad \textit{weakly in } H^{-1}\big(0,\tau;L^2_0(\Omega)\big) \tag{51}$$

and $(\widetilde{v}, p)$ is solution of (P). Furthermore $\dfrac{\partial \widetilde{v}}{\partial t}$ belongs to $L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big)$.

Proof. Recalling that the estimates (25)-(26) are independent of m, δ and ε, we deduce that $(\widetilde{v}_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2\big(0,\tau;\mathbf{H}^1(\Omega)\big) \cap L^\infty\big(0,\tau;\mathbf{L}^2(\Omega)\big)$. Moreover, since the estimate (39) is independent of δ and ε, the sequence $(p_\varepsilon)_{\varepsilon>0}$ is bounded in $H^{-1}\big(0,\tau;L^2(\Omega)\big)$ and we may infer the convergences (49)-(50)-(51). Furthermore the estimate (45) implies that that $\left(\dfrac{\partial \widetilde{v}_\varepsilon}{\partial t}\right)_{\varepsilon>0}$ is bounded in $L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big)$. Hence, possibly extracting another subsequence still denoted $(\widetilde{v}_\varepsilon)_{\varepsilon>0}$, we have

$$\frac{\partial \widetilde{v}_\varepsilon}{\partial t} \rightharpoonup \frac{\partial \widetilde{v}}{\partial t} \quad \text{weakly in } L^{\frac{4}{3}}\big(0,\tau;(\mathcal{V}_{0div})'\big)$$

and with the same arguments as in the previous Theorem, we get

$$\widetilde{v}_\varepsilon \to \widetilde{v} \quad \text{strongly in } L^2\big(0,\tau;\mathbf{L}^4(\Omega)\big),$$

$$\widetilde{v}_\varepsilon \to \widetilde{v} \quad \text{strongly in } L^2\big(0,\tau;\mathbf{L}^2(\Gamma_0)\big),$$

and

$$\widetilde{v}_\varepsilon^\delta \to \widetilde{v}_\varepsilon \quad \text{strongly in } \mathcal{C}^0(0,\tau;H),$$

for any Banach space H such that $\mathbf{L}^2(\Omega) \subset H \subset (\mathcal{V}_{0div})'$ with continuous injections and compact embedding of $\mathbf{L}^2(\Omega)$ into H.

With all these convergences and the assumption (15), we can pass to the limit in (47) and (48) by the same techniques as in Theorem 4.4 and Theorem 6.1 and we get (13) and (14). □ □

References

[1] J. Bemelmans, G.P. Galdi, M. Kyed, *Fluid flows around floating bodies, I: the hydrostatic case*, J. Math. Fluid Mech. 14(4) (2012) 751–770.

[2] M. Boukrouche, El Mir, *On the Navier-Stokes system in a thin film flow with Tresca free boundary condition and its asymptotic behavior*, Bull. Math. Soc. Sc. Math. Roumanie 48(96-2) (2005) 139–163.

[3] M. Boukrouche, L. Paoli, *Asymptotic analysis of a micropolar fluid flow in a thin domain with a free and rough boundary*, SIAM J. Math. Anal. 44(2) (2012) 1211–1256.

[4] M. Boukrouche, F. Saidi, *Non isothermal lubrication problem with Tresca fluid-solid interface law, part I*, Nonlinear Analysis Real World App. 7(5) (2006) 1145–1166.

[5] M. Boukrouche, I. Boussetouan, L. Paoli, *Existence for non-isothermal fluid flows with Tresca's friction and Cattano's heat law*, Submetted to Journal of Mathematical Analysis and applications.

[6] F. Boyer, P. Fabrie, Eléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles, Springer, Berlin, Heidelberg, 2000.

[7] E.A. Coddington, N. Levinson, Theory of ordinary differential equations, Mc Graw-Hill, New-York, 1955.

[8] G. Duvaut, J.L. Lions, Les inéquations en mécanique et physique, Dunod, Gauthiers-Villars, Paris, 1972.

[9] C. Foias, O. Manley, R. Rosa, R. Temam, Navier-Stokes equations and turbulence, Encyclopedia of mathematics and its applications 83, Cambridge University Press, 2001.

[10] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems, second ed., Springer Monographs in Mathematics, Springer, New York, 2011.

[11] V. Girault, P.A. Raviart, Finite element approximation of the Navier-Stokes equations, Springer-Verlag, Berlin, 1979.

[12] H. Hervet, L. Léger, *Flow with slip at the wall: from simple to complex fluids*, C. R. Acad. Sci. Paris, Physique 4 (2003) 241–249.

[13] V.A. Kondrat'ev, O.A. Oleinik, *Boundary-value problems for the system of elasticity theory in unbounded domains. Korn's inequailities*, Russian Math. Surveys 43(5) (1988) 65–119.

[14] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. 63(1) (1934) 193–248.

[15] J.L. Lions, *Some problems connected with Navier-Stokes equations*, Lectures at the IV^{TH} Latin-American School of Mathematics, Lima, July 1978.

[16] J.L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications, Springer-Verlag, New-York, Heidelberg, 1972.

[17] A. Biryuk, W. Craig, *Bounds on Kolmogorov spectra for Navier-Stokes equations*, Physica D: Nonlinear Phenomena 241(4) (2012) 426–438.

[18] P. Goulart, S. Chernyshenko, *Global stability analysis of fluid flows using sum-of-squares*, Physica D: Nonlinear Phenomena 241(6) (2012) 692–704.

[19] I.J. Rao, K.R. Rajagopal, *The effect of the slip boundary condition on the flow of fluids in a channel*, Acta Mechanica 135 (1999) 113–126.

[20] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Applic. 146 (1987) 65–96.

[21] J. Simon, *On the existence of the pressure for solutions of the variational Navier-Stokes equations*, J. Math. Fluid Mech. 1(3) (1999) 225–234.

[22] R. Temam, Navier-Stokes equations, theory and numerical analysis, Elsevier Science Publishers B.V, Amsterdam, 1984.

[23] J. Zhang, *An augmented Lagrangian approach to simulating yield stress fluid flows around a spherical gas bubble*, Internal. J. Numer. Methods Fluids 69(3) (2012) 731–746.

[24] I.Baffico and T. Sassi, *Existence result for a fluid structure interaction problem with friction type slip boundary condition.* ZAMM Z Angew. Math. Mech. 1–14 (2014)